NOTES

The Golden Section and the Piano Sonatas of Mozart

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Ubiquitous in nature, the golden section embodies its elegant proportion in the starfish and the chambered nautilus, in the pine cone and the sunflower, and in leaf patterns along the stems of plants [**11**, pp. 98, 113–114; **12**, pp. 3–14; **31**, pp. 150–169]. Perhaps it is because the golden section is in some sense natural, that artists, architects, and composers have often been influenced by it [**14**, **15**, **21**, **22**]. And perhaps this is to be expected, even when not deliberate, insofar as art imitates nature.

The *golden section* is defined to be that division of a line segment into two unequal segments such that the length a of the shorter segment is to the length b of the longer, as the length of the longer is to the whole. See FIGURE 1. That is,

$$\frac{a}{b} = \frac{b}{a+b}.$$

For convenience, we let the segment to be divided have length 1 and the shorter segment have length x as in FIGURE 2. Then the golden section is that division of the segment for which

$$\frac{x}{1-x} = \frac{1-x}{1}.$$

FIGURE 1
The golden section.

FIGURE 2

Solving this equation and setting φ equal to the common ratio, we have

$$\varphi = \frac{x}{1-x} = \frac{1-x}{1} = \frac{\sqrt{5}-1}{2} \approx 0.6180.$$

This ratio (or its reciprocal) is variously called the "golden ratio," "golden number," or "divine proportion," and is sometimes said to offer the most aesthetically pleasing proportion [**4**, p. 94; **16**, pp. 62–65; **36**, p. 74]. Whether or not this is accurate, the effect of identical ratios between the parts and between the parts and the whole is to unify the structure in a fundamental way.

At an age in excess of 24 centuries [**32**, p. 291], the golden section has become the subject of modern debate. In [**23**], for instance, G. Markowsky debunks some popularly held beliefs, such as, that the Parthenon, United Nations Building, and Great Pyramid conform to the golden section, and that rectangles with golden ratio proportions are the most pleasing. On the other hand, J. Benjafield and J. Adams-Webber [**2**] have advanced the "golden section hypothesis" that whenever people must divide a whole into two unequal parts, they tend to make the division near the golden section. In this paper, we take up the particular case of the piano sonatas of W. A. Mozart (1756–1791). Some have said that these works do reflect the golden section [**7**; **35**, p. 242]. Here, we analyze some collected data to judge whether there is convincing evidence to support this claim.

Even a listener who is only casually acquainted with the music of Mozart will hear something familiar in it; from Mozart's mind came melody not only delightful but memorable. On another level, the genius of the composer is manifested in form and balance. His music has been revered, among other things, for its "beautiful and symmetrical proportions" [**34**, p. 217]. In 1853, Henri Amiel opined that "the balance of the whole is perfect" [**1**, p. 54]. Hanns Dennerlein described Mozart's music as reflecting the "most exalted proportions," and the composer himself as having "an inborn sense for proportions" [quoted in **7**, p. 1], a thought echoed by H. C. Robbins Landon [**20**, p. 268]. Eric Blom wrote that Mozart had "an infallible taste for saying exactly the right thing at the right time and at the right length" [**5**, p. 265].

Music and mathematics having been happily entwined from antiquity, it is not surprising when talent in one accompanies enthusiasm for the other. Mozart's sister, Nannerl, recalled that when her brother was learning arithmetic, he gave himself entirely to it and that "he talked of nothing, thought of nothing but figures" [**19**, p. 124]. She recalled that he once covered the walls of the staircase and of all the rooms in their house with figures, then moved on to do the neighbors' houses as well. When he was 14, Mozart wrote to her asking that she send him arithmetical tables and more exercises in arithmetic [**25**, letter of April 21, 1770, p. 130, and letter of May 19, 1770, p. 137]. The margins of the manuscript of the *Fantasia and Fugue in C major* contain Mozart's calculations of the probability of winning a lottery [**18**, p. 178]. Alfred Einstein, one of Mozart's biographers wrote: "The pleasure of playing with figures remained with Mozart all his life long. Thus he once took up the problem, very popular at the time, of composing minuets 'mechanically,' by putting two-measure melodic fragments together in any order. And we possess a page of musical sketches on which he had begun to figure out the sum which the chess player would have received from the King in the famous Oriental story" [**9**, p. 25].

By the age of 18, Mozart had composed his first sonata for piano [**17**, p. 42; **29**, p. 45]. He wrote 19 altogether [**6**, pp. vi–vii], most of them during the next four years of his life, and almost all of them comprising three movements. In Mozart's time, the sonata-form movement was conceived in two parts [**26**, pp. 30–35; **28**, pp. 160, 163; **30**, pp. 1–2; **33**, pp. 14–15]: the Exposition in which the musical theme is introduced, and the Development and Recapitulation in which the theme is developed and revisited. See FIGURE 3. As a rule, each section was to be repeated (as indicated by the symbol :‖) in performance [**24**, p. xxxiii]. It is this separation into two distinct sections, together with the foregoing, which gives cause to wonder how Mozart apportioned these works.

FIGURE 3
Sonata-form movement.

Table 1 is a collection of data for all Mozart's sonata movements that are divided into two distinct sections, both of which are to be repeated in performance. Of the 56 movements, 29 are constructed in this way. The data are measure counts (taken from

[6]) of the lengths of the two sections: a represents the length of the Exposition, and b the length of the Development and Recapitulation. (When codas were present, they were not included as part of the second section.) The first column identifies the piece and movement by the Köchel cataloging system. K. 498a was not included because its authenticity is in doubt [**6**, p. vi; **29**, p. 56]. The first movement of the first sonata, K. 279, is 100 measures in length and is divided so that the Development and Recapitulation section has length 62. Note that 100φ, rounded to the nearest natural number, is 62. (These lengths are necessarily natural numbers because they are measure counts.) This is a perfect division according to the golden section in the following sense: A 100-measure movement could not be divided any closer (in natural numbers) to the golden section than 38 and 62. This is true of the second movement of this sonata as well. That is, a 74-measure movement cannot be divided any closer to the golden section than 28 and 46. Mozart did not, however, divide the third movement of K. 279

TABLE 1

Köchel	a	b	$a+b$
279, I	38	62	100
279, II	28	46	74
279, III	56	102	158
280, I	56	88	144
280, II	24	36	60
280, III	77	113	190
281, I	40	69	109
281, II	46	60	106
282, I	15	18	33
282, III	39	63	102
283, I	53	67	120
283, II	14	23	37
283, III	102	171	273
284, I	51	76	127
309, I	58	97	155
311, I	39	73	112
310, I	49	84	133
330, I	58	92	150
330, III	68	103	171
332, I	93	136	229
332, III	90	155	245
333, I	63	102	165
333, II	31	50	81
457, I	74	93	167
533, I	102	137	239
533, II	46	76	122
545, I	28	45	73
547a, I	78	118	196
570, I	79	130	209

exactly in golden section. An exact division would require b to be 98, not 102.

To evaluate the degree of consistency in these proportions, we use a scatter plot of b against $a+b$. If Mozart was consistent, there should be a linearity to the data, and if he divided movements near to the golden section, then the data points should fall near the line $y = \varphi x$. FIGURE 4 shows this scatter plot. Certainly, the linearity in the data is striking. The r^2 value is 0.990, confirming an extremely high degree of linearity. To this plot, we add two lines (FIGURE 5): the line $y = \varphi x$, and the regression line whose equation is $y = -0.003241 + 0.6091x$. The line $y = \varphi x$ scarcely differs from the line of best fit and, at the scale of FIGURE 5, it is difficult to see any difference at all between them. Of course, the line $y = \varphi x$ is a little above the regresssion line because of its slightly larger slope. Finally, a histogram (FIGURE 6) of the ratio $b/(a+b)$ reflects the centrality of φ.

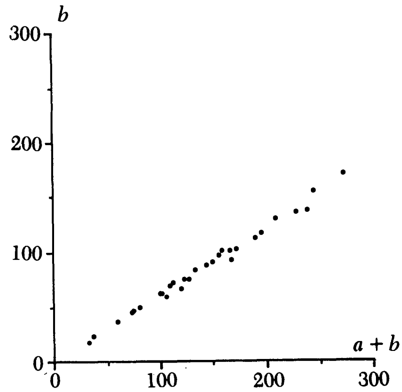

FIGURE 4

Scatter plot of b against $a+b$.

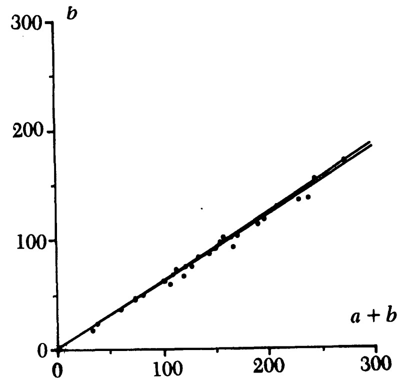

FIGURE 5

Scatter plot of b and $a+b$ with the line $y = \varphi x$ (top) and the regression line (bottom).

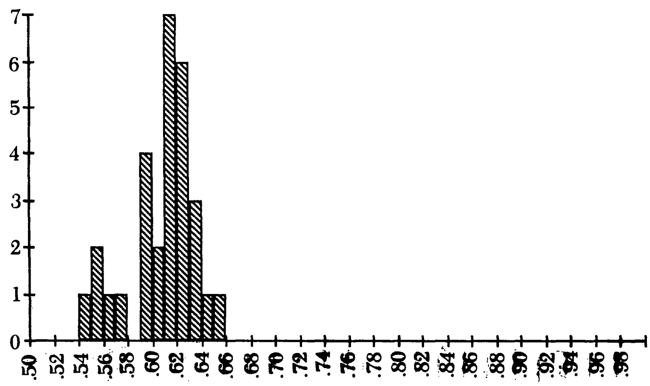

FIGURE 6

Frequency distribution of $\dfrac{b}{a+b}$.

This is impressive evidence that Mozart did, with considerable consistency, partition sonata movements near the golden section. Before we become convinced, however, let us analyze these data in another way. If a movement is divided in golden section, then both a/b and $b/(a+b)$ should be near φ. We have focused on $b/(a+b)$; let us concentrate now on a/b. FIGURE 7 is a scatter plot of a against b. Again, the data look very linear, though not so much so as b and $a+b$. To this plot, we again add the line $y = \varphi x$ and the regression line whose equation, this time, is $y = 1.360 + 0.6260x$. See FIGURE 8. The line of best fit, which is the one on the top in FIGURE 8, and the line $y = \varphi x$ again differ very little. The r^2 value of 0.938 verifies somewhat less goodness of fit. A histogram (FIGURE 9), however, is more revealing, showing much more variance than FIGURE 6 and, therefore, less evidence for the centrality of φ.

Now, why is this? It is possible, of course, to put a slant on the interpretation of data by the choice of presentation, but there seems to be something other than that going on here. And there is: It is a theorem **[10]** that what we have observed in these data is true for all data; $b/(a+b)$ is always nearer to φ than is a/b.

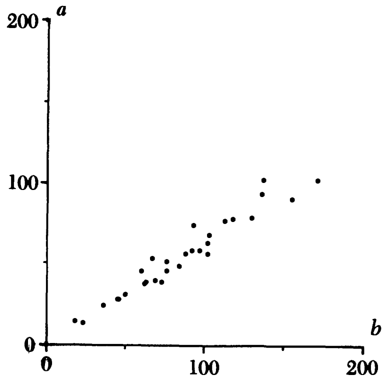

FIGURE 7
Scatter plot of a against b.

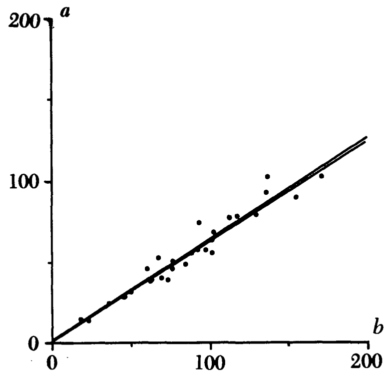

FIGURE 8
Scatter plot of a against b with the line $y=\varphi x$ (bottom) and the regression line (top).

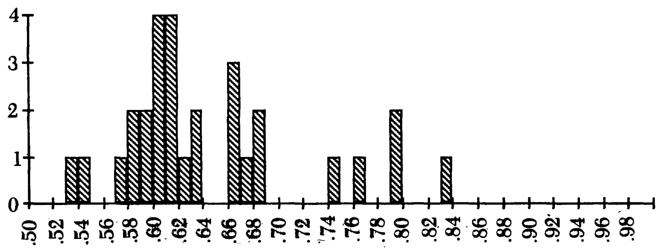

FIGURE 9
Frequency distribution of $\frac{a}{b}$.

THEOREM. $\left|\frac{b}{a+b}-\varphi\right| \le \left|\frac{a}{b}-\varphi\right|$ *where* $0 \le a \le b$.

Proof. Let $x=a/b$. Then we must show that

$$\left|\frac{1}{x+1}-\varphi\right| \le |x-\varphi|$$

for all $x \in [0,1]$. Let $f(x)=1/(x+1)$. By the Mean Value Theorem, for all $x \in [0,1]$ there is a $\xi \in (0,1)$ such that

$$|f(x)-f(\varphi)| = |f'(\xi)|\,|x-\varphi|.$$

Now $f'(x)=-1/(x+1)^2$ satisfies

$$\tfrac{1}{4} < |f'(x)| < 1$$

for $x \in (0,1)$. A simple calculation will show that φ is a fixed point of f, that is, that $f(\varphi)=\varphi$. So, for all $x \in [0,1]$,

$$\left|\frac{1}{x+1}-\varphi\right| \ge |x-\varphi|$$

with equality when $x=\varphi$.

We note, in passing, that the fixed-point algorithm of numerical analysis works on this principle, and that this theorem says that the ratio of consecutive terms of any Fibonacci-like sequence ($f_1=a, f_2=b, f_{n+2}=f_n+f_{n+1}$ with a and b not both zero) converges to φ.

Thus we know that, given *any* pair a and b, $0 \leqslant a \leqslant b$, the ratio $b/(a+b)$ will be closer to φ than a/b will. An enthusiast wishing to demonstrate a golden ratio relationship between, say, shoe size s and ACT score t, should present data in the form $t/(s+t)$ instead of s/t, because $t/(s+t)$ will be biased toward φ. Table 2 shows some data to demonstrate this. As the theorem predicts, in every case $t/(s+t)$ is nearer to 0.6180 than s/t is. One lesson to be learned from this is clear: If we are to analyze data in this way, then we must confine our investigations to the ratio a/b. Otherwise, we are likely to find φ even when it is not there. Unfortunately, some investigations have focused on the ratios $b/(a+b)$ **[2, 3, 8]**.

TABLE 2

Shoe size s	ACT score t	$\frac{t}{s+t}$	$\frac{s}{t}$
8	26	0.7647	0.3077
10	22	0.6875	0.4545
9	28	0.7568	0.3214
12	25	0.6757	0.4800

Drawing our attention, then, to a/b and following the ideas in **[10]**, let us ask what values we might reasonably expect the ratio to have. It would be absurd to think that any composer, at least in the classical period, would write, for example, a 200-measure sonata movement and divide it into two parts so lopsidedly as 1 and 199, or 2 and 198, or even 10 and 190. There simply would not be enough room in that to accomplish the purpose of the first section: the exposition of the theme. Quantz suggests some balance in the relationship in saying that "the first part must be somewhat shorter than the second" **[27**, p. 591**]**. So if we let the length of the movement $m = a + b$ be fixed, then a must be bounded below at some practical distance away from 0, and bounded above by $m/2$. For the moment, let us suppose that $m/4 \leq a \leq m/2$. This interval satisfies the conditions at least and has the appeal of simplicity. If we assume that a is randomly distributed, then an estimate of the expected value of a/b is

$$\begin{aligned} E(a/b) &\approx \frac{1}{m/4}\int_{m/4}^{m/2} \frac{x}{m-x}\,dx \\ &= \frac{4}{m}\left(x + m\ln|x-m|\right)\Big]_{m/2}^{m/4} \\ &= 4\ln\frac{3}{2} - 1 \\ &\approx 0.6219. \end{aligned}$$

This estimate differs from φ by about 0.6%. Of course, infinitely many other intervals also conform to the assumptions, and the expected values vary widely. For example, $0.3m \leqslant a \leq 0.4m$ gives $E(a/b) \approx 0.5415$. The data in Table 1 satisfy $0.348m \leq a \leq 0.455m$. Using this interval, $E(a/b) \approx 0.6753$. On the other hand, intervals satisfying the conditions can be chosen so that the expected value is exactly φ. One such interval is $[rm, (r+1/5)m]$ where

$$r = \frac{1-(4/5)e^{(\varphi+1)/5}}{1-e^{(\varphi+1)/5}}.$$

The point is this: The sonata form itself imposes restrictions. Depending on the assumptions we make about a to conform to them, these restrictions can induce on

a/b central tendency in the vicinity of φ and, in some cases, very near φ. And this is true even when a is determined, not by thoughtful design, but by uninspired randomness.

Still, we must remember that these sonatas *are* the work of a genius, and one who loved to play with numbers. Mozart may have known of the golden section and used it. That there is considerable deviation from it (FIGURE 9) suggests otherwise, however. Perhaps the golden section does, indeed, represent the most pleasing proportion, and perhaps Mozart, through his consummate sense of form, gravitated to it as the perfect balance between extremes. It is a romantic thought.

> *The "music" of nature and the music of man belong to two distinct categories. The transition from the former to the latter passes through the science of mathematics. An important and pregnant proposition. Still, we should be wrong were we to construe it in the sense that man framed his musical system according to calculations purposely made, the system having arisen through the unconscious application of pre-existent conceptions of quantity and proportion, through subtle processes of measuring and counting; but the laws by which the latter are governed were demonstrated only subsequently by science.*
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> —Eduard Hanslick, 1854 [**13**, p. 110]

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Can a Mathematician See Red?

Consider the sphere—
a hollow rounded surface
with no thickness.
Each point that we see
from the outside
is also a point we can see
from the inside.

If I paint red
all over the outside,
is the inside red?

The mathematician says NO,
for the layer of paint
forms a new sphere
that is outside the outside
and not a bit inside.

A mathematician
takes safe pleasure
in surface mysteries.

A poet
will see red
inside.

—JoAnne Growney
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